Asymptotic Properties of Energy of Harmonic Maps on Asymptotically Hyperbolic Manifolds

Man Chun LEUNG

Department of Mathematics, National University of Singapore
Singapore 119260   matlmc@nus.sg

Abstract

Asymptotic behavior of energy of a harmonic map defined on an asymptotically hyperbolic manifold is considered. Using the growth of energy, we show that a harmonic map defined on some asymptotically hyperbolic manifolds has to be constant if the total energy is finite, or if the map approaches a point fast enough, in terms of a defining function for the boundary.

KEY WORDS & AMS MS Classifications[1]

§1.   Introduction

Let (M, g) be a complete Riemannian manifold and (N, γ) be a Riemannian manifold. For a smooth map $f : M \to N$, the energy of the map f is given by

$$e(f) = g^{ij}\gamma_{\alpha\beta}(f)\frac{\partial f^{\alpha}}{\partial x_i}\frac{\partial f^{\beta}}{\partial x_j}\,.$$

Suppose that $f : M \to N$ is a harmonic map. A fundamental problem to consider is the relation between the harmonic map f and the energy $e(f)$. Burns, Garber, Ruijsenaars and Seiler [5] show that if (M, g) is the Euclidean space, then there are no non-constant harmonic maps with finite energy. As a special case of the results obtained in [13], Sealey shows that the same is true for the hyperbolic space. As often, if the total energy is infinite, then there is a growth formula on energy. Price [12] obtains the growth rate on energy for harmonic maps defined on the Euclidean space and Karcher and Wood [8] obtain the growth rate for simply connected complete Riemannian manifolds of bounded negative sectional curvature. The growth rate is used by Jin [7] to obtain the following interesting result. For $n \geq 3$, if $f : \mathbb{R}^n \to (N, \gamma)$ is a harmonic map such that $f(x) \to y_o \in N$ as $|x| \to \infty$, then f is a constant map.

---

[1]KEY WORDS: harmonic maps, asymptotically hyperbolic manifolds, energy estimate
 1991 AMS MS Classifications: Primary 58E20; Secondary 58G03

In this paper we study the energy behavior of of a harmonic maps defined on an asymptotically hyperbolic manifold. Asymptotically hyperbolic manifolds are generalizations of the hyperbolic space with the Poincaré metric. Other examples of asymptotically hyperbolic manifolds include some non-compact quotients of the hyperbolic space. They are studied by various authors in connection with the Hodge cohomology [9] and Einstein metrics with prescribed conformal structures at the boundary [6]. We consider a harmonic map which is defined outside a compact subset of an asymptotically hyperbolic manifold. The total energy is found to be either infinite or the energy decay exponentially in terms of the distance. An example shows that it is possible to have a finite energy harmonic map defined outside a compact subset of a hyperbolic space.

We study harmonic maps defined on the whole asymptotically hyperbolic manifold. It is shown that for a class of complete hyperbolic manifolds, there are no non-constant globally defined harmonic maps with finite energy. The growth rates of energy for harmonic maps on an asymptotically hyperbolic manifold are investigated. We show that if the total energy is infinite, then the energy grows exponentially in terms of the distance. The exponent can be chosen to be arbitrary close to that of the hyperbolic space. We note that previous results on growth rate of energy require the manifold to be simply connected.

It is well-known that hyperbolic space supports bounded harmonic maps (for example, see [2]). We use the asymptotic properties of energy to show that if a harmonic map defined on an asymptotically hyperbolic manifold approaches a fixed point fast enough, then the harmonic map has to be a constant. More precisely we prove the following result. For an integer $n \geq 3$, let $M = \mathbf{H}^{n+1}/\Gamma$ be a complete hyperbolic manifold without cusps and the exponent of convergence of the Poincaré series $\delta(\Gamma) < n/2 - 1$, or (M, g) is a simply connected asymptotically hyperbolic manifold with the sectional curvatures of (M, g) satisfying $-b^2 \leq K \leq -a^2$ with $na \geq 2b$. Let $f : (M, g) \to (N, \gamma)$ be a C^2-harmonic map and d_N be the distance function of N. If there exist a point $y_o \in N$ and positive constants C and σ such that

$$(1.1) \qquad d_N(y_o, f(x)) \leq C\rho^{1+\sigma}(x) \quad \text{for} \ \ x \in M \,,$$

then f is a constant map. If we assume that f can be extended to a C^2-map on the closure $\overline{M}$ (see section 1), then the same conclusion holds if

$$d_N(y_o, f(x)) \leq C\rho^\sigma(x) \,, \quad \text{for} \ \ x \in M \,.$$

The proof depends on a rather precise lower bound on the energy growth of a non-constant harmonic map defined on an asymptotically hyperbolic manifold. As in [7], the decay condition (1.1) gives an upper bound on the energy growth which contradicts with the lower bound unless the harmonic map is constant. We apply a regularity result to weaken the decay condition.

§ 2. Definitions and Basic Properties

We defined a class of Riemannian manifolds called conformally compact manifolds. Let M be the interior of a compact $(n+1)$-dimensional manifold $\overline{M}$ with boundary ∂M. Throughout this paper we assume that $n \geq 2$ unless otherwise is stated. A smooth function $\rho : \overline{M} \to R$ is said to be a defining function for ∂M if

$$\rho \geq 0, \quad \rho^{-1}(0) = \partial M \quad \text{and} \quad d\rho \neq 0 \quad \text{along} \quad \partial M.$$

A Riemannian metric g on M is said to be *conformally compact* if there exist a defining function ρ for ∂M and a Riemannian metric h on $\overline{M}$ such that $g = (1/\rho^2)h$. The Poincaré metric

$$(2.1) \qquad \qquad j_{ij} = \frac{4}{(1 - |x|^2)^2}\, \delta_{ij} \quad \text{for} \ \ x \in B^{n+1},$$

on the unit ball B^{n+1} is a conformally compact metric, with defining function

$$\rho_H(x) = \frac{1}{2}(1 - |x|^2) \quad \text{for} \ \ x \in B^{n+1}.$$

In general, (M, g) is a complete Riemannian manifold which may not be simply connected. Mazzeo [9] shows that the sectional curvatures of a conformally compact metric approach $-|\,d\rho\,|_h^2$ near the boundary ∂M. A conformally compact metric g is said to be *asymptotically hyperbolic* if $|\,d\rho\,|_h^2 = 1$ along the boundary. In this case, the sectional curvatures of an asymptotically hyperbolic metric approach -1 near the boundary. Let $x \in \partial M$ and $x' = (x_2, ..., x_{n+1})$ be a system of local coordinates in a neighborhood of p in ∂M. With $x_1 = \rho$, $(x_1, x_2, ..., x_{n+1})$ is a coordinate system in a neighborhood of p in $\overline{M}$. We may choose a defining function ρ such that in a neighborhood of ∂M, the metric g can be written as [6]

$$(2.2) \qquad \qquad g = \frac{1}{\rho^2}\left[d\rho^2 + \sum_{2 \leq i,\, j \leq n+1} h_{ij}(\rho,\, x')\, dx^i dx^j \right].$$

In the following we discuss some basic properties we need about harmonic maps. Let (M, g) and (N, γ) be Riemannian manifolds of dimension $n + 1$ and m respectively. Let $\{y_1, ..., y_m\}$ be a local coordinate system for N. If $f : M \to N$ is a C^2-map, then the energy density $e(f)$ of f is defined by

$$(2.3) \qquad \qquad e(f) = g^{ij}\gamma_{\alpha\beta}(f)\frac{\partial f^\alpha}{\partial x_i}\frac{\partial f^\beta}{\partial x_j}.$$

We observe the summation convention throughout this paper. The map f is called a harmonic map if f is a critical point of the energy functional

$$(2.4) \qquad \qquad E(f) = \int_M e(f)\, dg$$

with respect to compactly supported variations. Let f be a harmonic map. In local coordinates, f satisfies a system of equations [3]

$$(2.5) \qquad \Delta f^\alpha + g^{ij} \Gamma^\alpha_{\mu\nu} \frac{\partial f^\mu}{\partial x_i} \frac{\partial f^\nu}{\partial x_j} = 0 \,, \quad \alpha = 1, 2, ..., m \,,$$

where Δ is the Laplacian operator for (M, g) and $\Gamma^\alpha_{\mu\nu}$ are the Christoffel symbols of (N, γ).

As consequences of first variation, we have the following two formulas. The first one is with respect to a compactly supported vector field X on M. Let $f : M \to N$ be a harmonic map. We have [12]

$$(2.6) \qquad \int_M \{ e(f) \operatorname{div} X - 2 < df(\nabla_{e_i} X) \,, df(e_i) > \} \, dg = 0 \,,$$

where $\{e_1, ..., e_{n+1}\}$ is an orthonormal basis for $T_p M'$ and $<\,,\,>$ is the inner product of (N, γ). Assume that N is isometrically embedded into $\mathbf{R}^s$, for some big integer s. Let $\vartheta : M \to \mathbf{R}^s$ be a compactly supported map, then the first variational formula gives [7]

$$(2.7) \qquad \int_M g^{\alpha\beta} \left(2\gamma_{ij}(f) \frac{\partial f^i}{\partial x_\alpha} \frac{\partial \vartheta^j}{\partial x_\beta} + \frac{\partial \gamma_{ij}(f)}{\partial y_k} \vartheta^k \frac{\partial f^i}{\partial x_\alpha} \frac{\partial f^j}{\partial x_\beta} \right) dg = 0 \,.$$

The stress-energy tensor S_f of f is defined by

$$(2.8) \qquad S_f = \frac{1}{2} e(f) \cdot g - f^*(\gamma) \,.$$

From [3] we have

$$(2.9) \qquad (\operatorname{div} S_f)_i = g^{jk} \nabla_{e_j} S_{ki} = 0 \quad \text{for } i = 1, ..., n+1 \,.$$

Let D be a bounded domain in M with boundary ∂D and X be a smooth vector field on D, then [14]:

$$(2.10) \quad \frac{1}{2} \int_{\partial D} e(f) < X, \vec{\mathbf{n}} > ds = \int_D < S_f, \nabla X > dg + \int_{\partial D} < df(X), df(\vec{\mathbf{n}}) > ds \,,$$

where $\vec{\mathbf{n}}$ is the outward pointing unit normal vector field along ∂D and ds is the induced measure on ∂D. Therefore the integral $\int_D < S_f, \nabla X > dg$ depends only on the behavior of the vector field along the boundary. This is known as a conservation law for harmonic map.

§3. Asymptotic Properties of Energy

In this section we consider harmonic maps defined outside a compact subset of an asymptotically hyperbolic manifold (M, g). We start with an asymptotic result near the boundary of M. Let K be a compact set in M, which may be empty. Given a defining function ρ so that the Riemannian metric g can be written in the form as in 2.2. For a positive number σ, define

$$D_\sigma = \{x \in M \mid \rho(x) < \sigma\}.$$

If σ is small enough, then D_σ is a diffeomorpic to $\partial M \times (0, \sigma)$.

Theorem 3.1. *For an integer $n \geq 3$, let $f : (M \setminus K, g) \to (N, h)$ be a harmonic map. Then either $E(f) = \infty$ or, given any $\epsilon > 0$, there exist positive constants C and Υ such that*

$$\int_{D_\sigma} e(f)\, dg \leq C\sigma^{\ln(n-\epsilon)} \quad \text{for } 0 < \sigma < \Upsilon.$$

Proof. The defining function is chosen in such a way that g can be written in the form as in (2.2). Choose a positive number σ_o such that $K cap D_{\sigma_o} = \emptyset$. For $\sigma > 0$, let

$$H_\sigma = \{x \in M \mid \rho(x) = \sigma\}.$$

There exists a small positive number $\Upsilon < \sigma_o$ such that for all $0 < \sigma \leq \Upsilon$, H_σ is diffeomorphic to ∂M. Assume that $E(f) < \infty$. We need to show that

$$(3.2) \quad \int_\sigma^0 \left(\int_{H_\sigma} e(f)\, dS \right) \frac{d\rho}{\rho} = \int_\sigma^0 \left(\int_{H_\rho} e(f) \frac{dh}{\rho^n} \right) \frac{d\rho}{\rho} \leq C\sigma^{\ln(n-\epsilon)} \quad \text{for } \sigma < \Upsilon.$$

Here $dS = dh/\rho^n$ is the induced measure on H_σ by the conformally compact metric g and

$$dh = \sqrt{\det (h_{ij})_{2 \leq i,j \leq n+1}} \, dx_2 \cdots dx_{n+1}.$$

Denote

$$D_{\sigma_1, \sigma_2} = \{p \in M \mid \sigma_1 < \rho(p) < \sigma_2\}, \quad 0 < \sigma_1 < \sigma_2 < \Upsilon.$$

We have $\partial D_{\sigma_1, \sigma_2} = H_{\sigma_1} \cup H_{\sigma_2}$. $E(f) < \infty$ implies that

$$\int_\Upsilon^0 \left(\int_{H_\rho} e(f)\, dS \right) \frac{d\rho}{\rho} \leq E(f) < \infty.$$

Hence there exists a sequence real numbers $\{\sigma_i\}_{i=1}^\infty$ such that $\sigma_i < \Upsilon$ and

$$(3.3) \qquad \lim_{i \to \infty} \sigma_i = 0 \quad \text{and} \quad \lim_{i \to \infty} \int_{H_{\sigma_i}} e(f)\, dS = 0.$$

For any $\sigma < \Upsilon$ and i large, let

$$X' = -\rho \frac{\partial}{\partial \rho}$$

5

be a vector field defined on $D_{\sigma_i,\sigma}$. Applying (2.10) we obtain

$$\frac{1}{2}\int_{\partial D_{\sigma_i,\sigma}} e(f) <X',\vec{\mathbf{n}}> \, dS$$

$$= \int_{D_{\sigma_i,\sigma}} <S_f,\bigtriangledown X'> \, dg + \int_{\partial D_{\sigma_i,\sigma}} <df(X'), df(\vec{\mathbf{n}})> \, dS \, .$$

Since $\vec{\mathbf{n}} = \sigma\frac{\partial}{\partial\rho}$ along H_σ and $\vec{\mathbf{n}} = -\sigma_i\frac{\partial}{\partial\rho}$ along H_{σ_i}, we have

(3.4)
$$\frac{1}{2}\int_{H_{\sigma_i}} e(f) \, dS - \frac{1}{2}\int_{H_\sigma} e(f) \, dS$$

$$= \int_{D_{\sigma_i,\sigma}} <S_f,\bigtriangledown X'> \, dg + \int_{H_{\sigma_i}} <df(\vec{\mathbf{n}}), df(\vec{\mathbf{n}})> \, dS$$

$$- \int_{H_\sigma} <df(\vec{\mathbf{n}}), df(\vec{\mathbf{n}})> \, dS \, .$$

As

$$<df(\vec{\mathbf{n}}), df(\vec{\mathbf{n}})> \ \leq \ e(f) \, ,$$

if we let $i \to \infty$ and apply (3.3), then we have

(3.5)
$$\int_{H_\sigma} <df(\vec{\mathbf{n}}), df(\vec{\mathbf{n}})> \, dS = \int_{D_{0,\sigma}} <S_f,\bigtriangledown X'> \, dg + \frac{1}{2}\int_{H_\sigma} e(f) \, dS \, .$$

Using (2.2) and a calculation show that

$$\operatorname{div} X' = n - \frac{1}{2}\,\rho\, h^{kl}\frac{\partial h_{kl}}{\partial\rho} \, .$$

Let $x \in M$ with $\rho(x) < \sigma$. We may choose local coordinates about x such that

$$(e_1, e_2, ..., e_{n+1}) = (\rho\frac{\partial}{\partial\rho} , \rho\frac{\partial}{\partial x_2} , ..., \rho\frac{\partial}{\partial x_{n+1}})$$

is an orthonormal basis for $T_x(M)$. Using (2.2), we have

$$\bigtriangledown_{e_1} X' = 0 \quad \text{and} \quad \bigtriangledown_{e_j} X' = e_j - \frac{1}{2}\,\rho\, h^{kl}\frac{\partial h_{jl}}{\partial\rho}e_k \quad \text{for} \ \ 2 \leq j\,, k \leq n+1 \, .$$

From [13], we have

$$<S_f, \bigtriangledown X'> = \frac{1}{2}e(f) \operatorname{div} X' - <df(e_i), df(e_j)> <\bigtriangledown_{e_j} X', e_i> \, .$$

The calculation above shows that

$$<S_f, \bigtriangledown X'> \ = \ \frac{1}{2}e(f)\left[n - \frac{1}{2}\rho h^{kl}\frac{\partial h_{kl}}{\partial\rho}\right] + \frac{1}{2}\rho h^{il}\frac{\partial h_{jl}}{\partial\rho} <df(e_i), df(e_j)>$$

$$- \sum_{j=2}^{n+1} <df(e_j), df(e_j)> \, .$$

6

As $\rho \to 0$ on the boundary and h is a smooth metric on the closure $\overline{M}$, all the terms involving ρ can be bounded by $\frac{\delta}{2}e(f)$ for some positive constant δ. Therefore if we choose Υ to be small, then we have

(3.6) $\qquad < S_f, \nabla X' > \geq \frac{1}{2}e(f)(n-2-\delta) + < df(e_1), df(e_1) > \geq 0$

for $\rho(x) < \sigma < \Upsilon$. Furthermore δ can be taken to be depending on Υ and can be chosen to be as small as possible when $\Upsilon \to 0$. Then (3.5) and (3.6) give

(3.7) $\qquad \int_{H_\sigma} < df(\vec{\mathbf{n}}), df(\vec{\mathbf{n}}) > dS \geq \frac{1}{2}\int_{H_\sigma} e(f)\,dS \quad$ for $\sigma < \Upsilon$.

Equity holds if and only if

$$\int_{D_{0,\sigma}} < S_f, \nabla X' > dg = 0\,.$$

From (3.6), this is the case if and only if $e(f) \equiv 0$ on $D_{0,\sigma}$. By unique continuation, this is equivalent to f being a constant map.

We introduce a change of variables $e^{-t} = \rho$ so that $e^{-\tilde{t}} = \Upsilon$ for some large number $\tilde{t}$. We have

$$\int_\Upsilon^0 \left(\int_{H_\rho} e(f)\,dS \right) \frac{d\rho}{\rho} = \int_{\tilde{t}}^\infty \left(\int_{H_{e^{-t}}} e(f)\,dS \right) dt < \infty\,.$$

We want to show that for any positive constant ϵ, there exists a positive constant $t_o > \tilde{t}$ such that for all $t_1 > t_o$ we have

(3.8) $\qquad \int_{t_1}^{t_1+1} \left(\int_{H_{e^{-t}}} e(f)\,dS \right) dt \geq (n-1-\epsilon) \int_{t_1+1}^\infty \left(\int_{H_{e^{-t}}} e(f)\,dS \right) dt\,.$

Assume that (3.8) is not true. Then there exists a positive constant $\epsilon > 0$ such that no matter how large t_o is, there exists a $t_1 > t_o$ such that

(3.8′) $\qquad \int_{t_1}^{t_1+1} \left(\int_{H_{e^{-t}}} e(f)\,dS \right) dt < (n-1-\epsilon) \int_{t_1+1}^\infty \left(\int_{H_{e^{-t}}} e(f)\,dS \right) dt\,.$

Let $t_2 > t_1$ and

$$X = \chi(t) \frac{\partial}{\partial t}$$

be a vector field defined on $D_{e^{-t_2}, e^{-t_1}}$, where χ is defined by

$$\chi(t) = \begin{cases} 0, & \text{if } t \leq t_1; \\ t - t_1, & \text{if } t_1 \geq t \leq t_1 + 1\,; \\ 1, & \text{if } t_2 \geq t \geq t_1 + 1\,. \end{cases}$$

A calculation similar to (3.6) gives, for $t \in [t_1, t_1 + 1]$,

$$
\begin{aligned}
< S_f, \triangledown X > \ = \ & \frac{1}{2}e(f)\left[n(t - t_1) + 1 - \frac{1}{2}(t - t_1)\rho\, h^{kl}\, \frac{\partial h_{kl}}{\partial \rho}\right] \\
& + \frac{1}{2}(t - t_1)\rho\, h^{kl}\, \frac{\partial h_{jl}}{\partial \rho} < df(e_k), df(e_j) > \\
& - \sum_{j=2}^{n+1}(t - t_1) < df(e_j), df(e_j) > - < df(e_1), df(e_1) > \\
\geq \ & \frac{1}{2}e(f)\left[(n - 2 - \delta)(t - t_1)\right] + (t - t_1) < df(e_1), df(e_1) > \\
& - < df(e_1), df(e_1) > + \frac{1}{2}e(f) \\
\geq \ & -\frac{1}{2}e(f)\,.
\end{aligned}
$$

For $t \in [t_1 + 1, t_2]$, we have

$$
\begin{aligned}
< S_f, \triangledown X > \ = \ & \frac{1}{2}e(f)\left[n - \frac{1}{2}\rho\, h^{kl}\, \frac{\partial h_{kl}}{\partial \rho}\right] \\
& + \frac{1}{2}\rho\, h^{kl}\, \frac{\partial h_{jl}}{\partial \rho} < df(e_k), df(e_j) > - \sum_{j=2}^{n+1} < df(e_j), df(e_j) > \\
\geq \ & \frac{1}{2}e(f)[(n - 2 - \delta)] + < df(e_1), df(e_1) >\,.
\end{aligned}
$$

Using (2.10) and (3.7) we obtain

$$
\begin{aligned}
(3.9) \qquad \frac{1}{2}\int_{H_{e^{-t_2}}} e(f)\, dS = \ & \int_{t_1}^{t_1+1}\left(\int_{H_{e^{-t}}} < S_f, \triangledown X > dS\right) dt \\
& + \int_{t_1+1}^{t_2}\left(\int_{H_{e^{-t}}} < S_f, \triangledown X > dS\right) dt + \int_{H_{e^{-t_2}}} < df(e_1), df(e_1) > dS \\
\geq \ & -\frac{1}{2}\int_{t_1}^{t_1+1}(\int_{H_{e^{-t}}} e(f)\, dS)dt \\
& + \frac{1}{2}(n - 1 - \delta)\int_{t_1+1}^{t_2}\left(\int_{H_{e^{-t}}} e(f)\, dS\right) dt + \int_{H_{e^{-t_2}}} < df(e_1), df(e_1) > dS\,.
\end{aligned}
$$

Given any $\varepsilon' > 0$, we can choose t_2 large enough such that

$$
(1 + \varepsilon')\int_{t_1+1}^{t_2}\left(\int_{H_{e^{-t}}} e(f)\, dS\right) dt \geq \int_{t_1+1}^{\infty}\left(\int_{H_{e^{-t}}} e(f)\, dS\right) dt\,.
$$

If we choose δ and ε' to be small, that is, if we choose t_o to be large, then we have

$$
n - 1 - \epsilon \ \leq \ \frac{n - 1 - \delta}{1 + \varepsilon'}\,.
$$

(3.8′) and (3.9) give

$$\frac{1}{2}\int_{H_{e^{-t_2}}} e(f)\,dS$$

$$> \quad -\frac{1}{2}\int_{t_1}^{t_1+1}\left(\int_{H_{e^{-t}}} e(f)\,dS\right)dt + \frac{1}{2}\left(\frac{n-1-\delta}{1+\varepsilon'}\right)\int_{t_1+1}^{\infty}\left(\int_{H_{e^{-t}}} e(f)\,dS\right)dt$$

$$+\int_{H_{e^{-t_2}}} <df(e_1),df(e_1)>\,dS$$

$$\geq \quad \int_{H_{e^{-t_2}}} <df(e_1),df(e_2)>\,dS\,.$$

The above inequality is in contradiction with (3.7). We note that by letting t_o to be large, we can choose δ and ε' to be as small as we like, that is, we can choose ϵ as small as we like. Then (3.8) holds for all t_1 large. For $t > t_1$ let

$$F(t) = \int_t^{\infty}\left(\int_{H_{e^{-t}}} e(f)\,dS\right)dt\,.$$

(3.8) implies that

$$F(t) - F(t+1) \geq (n-1-\epsilon)F(t+1)\,,$$

when t is large. That is,

$$F(t) \geq (n-\epsilon)F(t+1) \geq \cdots \geq (n-\epsilon)^{m-1}F(t+m)\,.$$

There exists a positive constant C such that

(3.10) $$F(t) \leq C(n-\epsilon)^{-t} = Ce^{-t\ln(n-\epsilon)} \quad \text{for } t \text{ large}\,.$$

Using the relation $e^{-t} = \rho$ we have obtained (3.2). (3.2) or (3.10) indicate an exponential decay in terms of the distance to a fixed point in (M,g). □

Remark 3.11. There are non-constant harmonic maps defined outside a compact set of a hyperbolic space which have finite total energy. Let $r = |x|$, where $x \in B^{n+1}$. Then

$$u(r) = \int_r^1 \frac{(1-r^2)^{n-2}}{r^{n-1}}dt$$

is a non-constant harmonic function defined on $B^{n+1}\setminus\{0\}$. f is singular at 0 [1]. The energy density is given by

$$e(u) = \rho_H^2\left|\frac{(1-t^2)^{n-2}}{t^{n-1}}\right|^2 = O(\rho_H^{2n-2})\,.$$

Hence the total energy inside D_{ρ_H} decays in order $O(\rho_H^{n-2})$.

Let M be an asymptotically hyperbolic manifold. Let f be a harmonic map

9

defined on the *whole M*. The above example demonstrates that we need to use a global argument in order to conclude that f is a constant map if $E(f) < \infty$. We consider the most interesting case when M is a hyperbolic manifold. Let Γ be a discrete group of isometries on $\mathbf{H}^{n+1}$. Suppose that Γ is without parabolic elements and $M = \mathbf{H}^{n+1}/\Gamma$ is a complete non-compact manifold. It is known that M is the interior of a compact manifold and the metric is conformally compact [11]. Let $\delta(\Gamma)$ be the exponent of convergence of the Poincaré series corresponding to Γ. Then for any $\alpha > \delta(\Gamma)$, the series

$$\sum_{\gamma \in \Gamma} e^{-\alpha d_H(x, \gamma x)}$$

converges, where $x \in \mathbf{H}^{n+1}$ is a fixed point and d_H is the distance function in the hyperbolic space.

Lemma 3.12.   *For an integer $n \geq 3$, let $M = \mathbf{H}^{n+1}/\Gamma$ be a complete hyperbolic manifolds with $\delta(\Gamma) < n/2 - 1$. Suppose that Γ has no parabolic elements. Then there is no non-constant harmonic map on M with finite energy.*

Proof.   Let α be a number such that

$$\delta(\Gamma) < \alpha < \frac{n}{2} - 1 \,.$$

We have

$$(3.13) \qquad \sum_{\gamma \in \Gamma} e^{-\alpha d_H(x, \gamma x)} \geq \sum_{k=0}^{\infty} \sharp\{\gamma \in \Gamma \mid k \ < \ d_H(x, \gamma x) \leq k + 1\} \, e^{-(k+1)\alpha} \,.$$

The left hand side of (3.13) converges by the comparison test. Therefore we have

$$\lim_{k \to \infty} \sharp\{\gamma \in \Gamma \mid k < d_H(x, \gamma x) \leq k + 1\} \, e^{-(k+1)\alpha} = 0 \,.$$

Hence there exists a positive constant C such that

$$\sharp\{\gamma \in \Gamma \mid k < d_H(x, \gamma x) \leq k + 1\} \leq C e^{-(k+1)\alpha} \,.$$

We obtain

$$(3.14) \qquad \sharp\{\gamma \in \Gamma \mid 0 < d_H(x, \gamma x) \leq k\} \leq C \sum_{i=0}^{k} e^{-(i+1)\alpha} \leq C' \int_0^k e^{\alpha x} dx \leq C'' e^{\alpha k} \,,$$

where C' and C'' are positive constants. Let $B_x(k) \subset B_x(2k)$ be balls in $\mathbf{H}^{n+1}$ with center at x and radius k and $2k$ respectively, where $k = 1, 2, \ldots$. Let $p : \mathbf{H}^{n+1} \to M$ be the covering map. Assume that there is a harmonic map $f : M \to N$ with finite energy. Let C_o be a positive number such that $E(f) < C_o$. Denote $\tilde{f}$ the lifting of f to $\mathbf{H}^{n+1}$. For each $y \in B_x(k)$, we have $d_M(p(y), p(x)) \leq k$, where d_M is the distance in M. Hence there exists a fundamental domain Δ such that y and γx are

inside Δ for some $\gamma \in \Gamma$. Furthermore $d_H(y, \gamma x) \leq k$. By (3.14), the number of γx inside $B_x(2k)$ is lesser than or equal to $C'' e^{2\alpha k}$. Therefore the number of fundamental domains needed to cover $B_x(k)$ is lesser than or equal to $C'' e^{2\alpha k}$. Let $\tilde{E}(\tilde{f}, k)$ be the total energy of $\tilde{f}$ on $B_x(k)$. Since $E(f) < C_o$, we have

$$\tilde{E}(\tilde{f}, k) \leq C_o C'' e^{2\alpha k} .$$

If $\tilde{f}$ is a non-constant harmonic map, then $\tilde{E}(\tilde{f}, k)$ grows in the order of $e^{(n-2)k}$ [8]. If $\alpha < n/2 - 1$, then $\tilde{f}$ has to be a constant map. Hence f is a constant map. $\qquad\square$

§4.   Energy Growth and Liouville Type Theorem

We show that if f is a harmonic map defined on the whole of M and has infinite energy, then the energy grows exponentially in terms of the distance.

Theorem 4.1.   *For an integer $n \geq 3$, let (M, g) be an asymptotically hyperbolic manifold. If $f : (M, g) \to (N, \gamma)$ is a harmonic map with $E(f) = \infty$, then for any $\delta > 0$, we have*
$$E(f, \varrho) = \int_{\varrho}^{\infty} \left(\int_{H_\sigma} e(f) \, dS \right) \frac{d\rho}{\rho} \geq \frac{C}{\varrho^{n-2-\delta}} ,$$
where C is a positive constant depending on f and δ.

Proof.   Let
$$X = \xi(\rho) \cdot \rho \frac{\partial}{\partial \rho} ,$$

where ξ is a compactly supported function to be determined later. A calculation using (2.2) shows that

$$\operatorname{div} X = \rho \frac{d\xi}{d\rho} - n\xi + \frac{1}{2} \xi \, \rho \, h^{kl} \frac{\partial h_{kl}}{\partial \rho} .$$

Let $x \in M$. We may choose local coordinates above x such that

$$(e_1, e_2, \ldots, e_{n+1}) = (\rho \frac{\partial}{\partial \rho}, \rho \frac{\partial}{\partial x_2}, \ldots, \rho \frac{\partial}{\partial x_{n+1}})$$

is an orthonormal basis for $T_x(M)$. Then

$$\nabla_{e_1} X = (\rho \xi') \, e_1 \quad \text{and} \quad \nabla_{e_j} X = \left(\frac{1}{2} \xi \, \rho \, h^{kl} \frac{\partial h_{jl}}{\partial \rho} \right) e_k - \xi e_j$$

for $2 \leq j, k \leq n+1$. Using formula (2.6), we obtain

(4.2) $\quad 0 \;=\; \int_M e(f) \left\{ \rho \xi' - n\xi + \frac{1}{2} \xi \rho h^{kl} \frac{\partial h_{kl}}{\partial \rho} \right\} dg$

$$-2 \int_M \left\{ \rho \xi' < df(e_1), df(e_1) > - \sum_{j=2}^{n+1} \xi < df(e_j), df(e_j) > \right\} dg$$

$$- \int_M \rho \, \xi \, h^{kl} \frac{\partial h_{jl}}{\partial \rho} < df(e_k), df(e_j) > \, dg \, .$$

Given any positive number δ, using the fact that

$$e(f) = \sum_{j=1}^{n+1} < df(e_j), df(e_j) > \, ,$$

and (4.2), we have

(4.3) $\quad - \int_M \rho \, \xi' \, e(f) \, dg + (n - 2 - \delta) \int_M \xi e(f) \, dg$

$$= \; -\delta \int_M \xi e(f) \, dg - 2 \int_M \left\{ \rho \xi' < df(e_1), df(e_1) > \; + \xi < df(e_1), df(e_1) > \right\} dg$$

$$+ \frac{1}{2} \int_M \xi \, \rho \, h^{kl} \frac{\partial h_{kl}}{\partial \rho} e(f) \, dg - \int_M \rho \, \xi \, h^{kl} \frac{\partial h_{jl}}{\partial \rho} < df(e_k), df(e_j) > \, dg \, .$$

Given $\epsilon > 0$, let

$$\phi(x) = \begin{cases} 1, & \text{if } t \le 1; \\ 1 + \frac{1-t}{\epsilon}, & \text{if } 1 < t < 1 + \epsilon; \\ 0, & \text{if } t \ge 1 + \epsilon. \end{cases}$$

For $\tau_1 > \tau > 0$, set

$$\xi(\rho) = \phi\left(\frac{\tau}{\rho}\right) \cdot \left(1 - \phi\left(\frac{\tau_1}{\rho}\right)\right) \, .$$

Then

(4.4) $$\rho \frac{\partial \xi}{\partial \rho} = -\tau \frac{\partial \xi}{\partial \tau} - \tau_1 \frac{\partial \xi}{\partial \tau_1} \, .$$

Substitute into (4.3), we obtain

(4.5) $\quad \tau \frac{\partial}{\partial \tau} \left(\int_M \xi e(f) \, dg \right) + (n - 2 - \delta) \int_M \xi e(f) \, dg$

$$= \; -\delta \int_M \xi e(f) \, dg - \tau_1 \frac{\partial}{\partial \tau_1} \left(\int_M \xi e(f) \, dg \right) + 2\tau \frac{\partial}{\partial \tau} \left(\int_M \xi < df(e_1), df(e_1) > \, dg \right)$$

$$+ 2\tau_1 \frac{\partial}{\partial \tau_1} \left(\int_M \xi < df(e_1), df(e_1) > \, dg \right) - 2 \int_M \xi < df(e_1), df(e_1) > \, dg$$

$$+ \frac{1}{2} \int_M \xi \, \rho \, h^{kl} \frac{\partial h_{kl}}{\partial \rho} e(f) \, dg - \int_M \rho \, \xi \, h^{kl} \frac{\partial h_{jl}}{\partial \rho} < df(e_k), df(e_j) > \, dg \, .$$

Since h is a smooth metric on closure $\overline{M}$, there exists a positive constant C_o such that

$$|\rho \, h^{kl} \frac{\partial h_{kl}}{\partial \rho}| \le C_o \rho \quad \text{and} \quad |\rho \, h^{kl} \frac{\partial h_{jl}}{\partial \rho} < df(e_k), df(e_j) > | \le C_o \rho e(f) \, .$$

12

For the positive number δ, if τ_1 and τ are small, then the last two terms in the right hand side of (4.5) are dominated by the first term. Consider the second and third term in the right hand side of (4.5), they are negative. As for the fourth term, it is independent of τ. If $E(f) = \infty$, then $E(f, \tau) \to \infty$ as $\tau \to 0$. Therefore the fourth term is also dominated by the first term if τ is small enough. Finally, the fifth term is negative. Hence

$$(4.6) \qquad \frac{\partial}{\partial \tau} \left(\tau^{n-2-\delta} \int_M \xi e(f) \, dg \right) \leq 0 \,,$$

which is ture for τ small. Let $\epsilon \to 0$ and then integrate, we have

$$E(f, \rho) \geq \frac{C}{\rho^{n-2-\delta}} \,,$$

where C is a positive constant depending on the map f and the constant δ. $\qquad \square$

It is shown in [5] and [12] that if $f : \mathbf{R}^n \to N$ is a non-constant harmonic map, then $E(f)$ cannot be finite. Moreover, [12] contains a growth formula on energy for Euclidean spaces. Later, in [8] and [13], it is showed that if f is a non-constant harmonic map defined on a complete, simply connected manifold of suitably pinched negative sectional curvature, then $E(f)$ again cannot be finite. Also, a growth formula is obtained in [8]. Except for the arbitrary positive constant δ, the growth formula in [8] agrees with the present formula on hyperbolic spaces. For further discussion on finiteness and growth of energy, we refer to the comprehensive surveys [3] and [4], and the references within.

Remark 4.7. By lemma 3.12 and theorem 4.1, if $M = \mathbf{H}^{n+1}/\Gamma$ is a complete hyperbolic manifold without cusps and $\delta(\Gamma) < n/2 - 1$, then any non-constant harmonic on M have the energy growth specified in theorem 4.1.

Remark 4.8. If (M, g) is a *simply connected* asymptotically hyperbolic manifold with negative sectional curvature. Suppose that the sectional curvatures of (M, g) satisfy $-b^2 \leq K \leq -a^2$ with $na \geq 2b$. Then any harmonic map with finite total energy is a constant map (cf. [8]). Hence any non-constant harmonic map on (M, g) have energy growth as in theorem 4.1.

Theorem 4.9. *For an integer $n \geq 3$, let M be a asymptotically hyperbolic manifold and $f : M \to N$ be a harmonic map. If there exists a point $y_o \in N$ such that*

$$d_N(f(x), y_o) \leq C\rho^{1+\sigma}(x) \quad \text{for} \ \ x \in M \,,$$

where $\sigma > 0$ and d_N is the distant function of N, then f has finite energy. In particular, if $M = \mathbf{H}^{n+1}/\Gamma$ is a complete hyperbolic manifold without cusps and $\delta(\Gamma) <

$n/2 - 1$, or (M, g) is a simply connected asymptotically hyperbolic manifold with the sectional curvatures of (M, g) satisfying $-b^2 \leq K \leq -a^2$ with $na \geq 2b$, then f is a constant map.

Proof. Let $\{y_1, ..., y_m\}$ be a local coordinate system above a neighborhood U of y_o such that $y_o = 0$ and

(4.10) $$\frac{\partial \gamma_{ij}(y)}{\partial y_k} y_k + 2\gamma_{ij}(y) \geq \gamma_{ij}(y) \quad \text{for } y \in U .$$

See [7] for a discussion on the existence of such a local coordinate neighborhood. In such a local coordinate system we have

$$|f^i(x)| = |y_i \circ f(x)| \leq C\rho^{1+\sigma}(x) \quad \text{for } x \in M .$$

Near ∂M, let $\{x_1 = \rho, x_2, ..., x_{n+1}\}$ be local coordinates for M, where ρ is a defining function such that the Riemannian metric g can be expressed in the form as in (2.2). Then

$$E(f) = \int_M \rho^2 h^{\alpha\beta}\gamma_{ij}\frac{\partial f^i}{\partial x_\alpha}\frac{\partial f^j}{\partial x_\beta}\frac{1}{\rho^{n+1}}\, dh .$$

Let ξ be defined as in the proof of theorem 4.1. The first variational formula (2.7) with respect to

$$\vartheta(x) = \xi(\rho(x))f$$

gives (cf. [7])

(4.11) $$\int_M \left[\frac{\partial \gamma_{ij}(f)}{\partial y_k}f^k + 2\gamma_{ij}(f)\right] h^{\alpha\beta}\frac{\partial f^i}{\partial x_\alpha}\frac{\partial f^j}{\partial x_\beta}\xi(\rho(x))\frac{1}{\rho^{n-1}}\, dh$$
$$= -2\int_M h^{\alpha\beta}\gamma_{ij}\frac{\partial f^i}{\partial x_\alpha}f^j\frac{\partial \xi}{\partial x_\beta}\frac{1}{\rho^{n-1}}\, dh .$$

We have

(4.12) $$\frac{\partial \xi}{\partial x_\beta} = \frac{1}{\epsilon\tau} \cdot \frac{\tau^2}{\rho^2}\frac{\partial \rho}{\partial x_\beta} \quad \text{for } (1 + \epsilon)\tau > \rho > \tau .$$

If we let $\epsilon \to 0$, we obtain

(4.13) $$\int_{M(\tau)\backslash M(\tau_1)} \left[\frac{\partial \gamma_{ij}(f)}{\partial y_k}f^k + 2\gamma_{ij}(f)\right] h^{\alpha\beta}\frac{\partial f^i}{\partial x_\alpha}\frac{\partial f^j}{\partial x_\beta}\frac{1}{\rho^{n-1}}\, dh + R(\tau_1)$$
$$= -2\int_{\partial M(\tau)} h^{\alpha\beta}\gamma_{ij}\frac{\partial f^i}{\partial x_\alpha}f^j\frac{\partial \rho}{\partial x_\beta}\frac{1}{\rho^{n-1}}\, ds ,$$

where $\tau_1 > \tau$ and $R(\tau_1)$ is a term depending on τ_1 but not on τ. Here ds is the measure of ∂M_τ induced by the Riemannian metric h. Recall that

$$M_\tau = \{ x \in M \mid \rho(x) > \tau \} .$$

Let

(4.14)
$$F(\tau) = \int_{M(\tau)\setminus M(\tau_1)} \gamma_{ij}(f) h^{\alpha\beta} \frac{\partial f^i}{\partial x_\alpha} \frac{\partial f^j}{\partial x_\beta} \frac{1}{\rho^{n-1}} \, dh + R(\tau_1) \, .$$

Differentiating both side of (4.14) we have

$$\frac{dF}{d\tau} = -\int_{\partial M(\tau)} \gamma_{ij}(f)) h^{\alpha\beta} \frac{\partial f^i}{\partial x_\alpha} \frac{\partial f^j}{\partial x_\beta} \frac{1}{\rho^{n-1}} \, ds \, .$$

Using Hölder's inequality and the fact that ρ and $h^{\alpha\beta}$ are smooth functions on the closure $\overline{M}$, we obtain

(4.15)
$$\left(-\int_{\partial M(\tau)} h^{\alpha\beta} \gamma_{ij} \frac{\partial f^i}{\partial x_\alpha} f^j \frac{\partial \rho}{\partial x_\beta} \frac{1}{\rho^{n-1}} \, ds \right)^2$$
$$\leq \ C' \left(\int_{\partial M(\tau)} \gamma_{ij}(f)) h^{\alpha\beta} \frac{\partial f^i}{\partial x_\alpha} \frac{\partial f^j}{\partial x_\beta} \frac{1}{\rho^{n-1}} \, ds \right) \times \left(\int_{\partial M(\tau)} \gamma_{ij} f^i f^j \frac{1}{\rho^{n-1}} \, ds \right) ,$$

where C' is a positive constant depending on ρ and h. Since $d_N(f(x), y_o) = O(\rho^{1+\sigma}(x))$ for some $\sigma > 0$, we have

(4.16)
$$\int_{\partial M(\tau)} h_{ij} f^i f^j \frac{1}{\rho^{n-1}} \, ds \leq \frac{C''}{\tau^{n-3-2\sigma}} \, ,$$

for some constant $C'' > 0$. Using (4.14), (4.15) and (4.16), we obtain

$$F^2(\tau) \leq -\frac{C''}{\tau^{n-3-2\sigma}} \frac{dF}{d\tau} \, ,$$

or

(4.17)
$$\frac{d}{d\tau}\left(\frac{1}{F(\tau)} \right) \geq \frac{\tau^{n-3-\sigma}}{C''} \, .$$

Assume that the harmonic map f has infinite energy. Then $F(\tau) \to \infty$ as $\tau \to 0$. Hence

$$\frac{1}{F(\tau)} = \int_0^\tau \frac{d}{d\tau}\left(\frac{1}{F(\tau)} \right) d\tau \geq \frac{1}{C''} \int_0^\tau \tau^{n-3-2\sigma} \, d\tau \, .$$

Or

(4.18)
$$E(f, \tau) \leq C'' \tau^{-(n-2-2\sigma)} \, .$$

As the total energy is infinity, we can apply theorem 4.1. If we choose the positive constant δ in theorem 4.1 to be lesser than 2σ, then we have obtained a contradiction. Hence the total energy of f is finite. In case $M = \mathbf{H}^{n+1}/\Gamma$ is a complete hyperbolic manifold without cusps and $\delta(\Gamma) < n/2 - 1$, or (M, g) is a simply connected asymptotically hyperbolic manifold with the sectional curvatures of (M, g) satisfying

15

$-b^2 \leq K \leq -a^2$ with $na \geq 2b$, then f is a constant map by remark 4.7 and 4.8. $\square$

We introduce normal coordinates $\{r, \omega_2, ...\omega_m\}$ about y_o, where r is the radial distance to q and $(\omega_2, ...\omega_m) \in S^{m-1}$. Denote

$$f^1 = r \circ f, \ f^2 = \omega_2 \circ f, ..., \ f^m = \omega_m \circ f.$$

Lemma 4.19. *Let M be an asymptotically hyperbolic manifold and $f : M \to N$ be a harmonic map. Assume that $f^i, 2 \leq i \leq m$, can be extended as C^2-functions on $\overline{M}$. If there exists a point $y_o \in N$ such that*

$$d_N(f(x), y_o) \leq C\rho^\nu(p) \quad \text{for some} \ \ \nu > 0, ,.$$

then

$$d_N(f(p), q) = O(\rho^n).$$

Proof.   As above, $\{x_1 = \rho, x_2, ..., x_{n+1}\}$ is a local coordinate system for M. The equations for harmonic maps are of the form

(4.20)
$$J(f^1) = \Delta f^1 + \rho^2 h^{\alpha\beta} \Gamma^1_{ij} \frac{\partial f^i}{\partial x_\alpha} \frac{\partial f^j}{\partial x_\beta} = 0.$$

We regard this as an equation on f^1. The terms involving f^i and its derivatives, for $2 \leq i \leq m$, are considered to be coefficients of the equation. By the assumption we have $f^1 = O(\rho^\nu)$. Let L be the linearization of J about the zero function, that is,

(4.21) $$L(v) = \Delta v + \rho^2 h^{\alpha\beta} \left(\Gamma^1_{1j} \frac{\partial v}{\partial x_\alpha} \frac{\partial f^j}{\partial x_\beta} + \Gamma^1_{i1} \frac{\partial f^i}{\partial x_\alpha} \frac{\partial v}{\partial x_\beta} \right), \quad 2 \leq i, j \leq m.$$

The normal operator [10] of L is Δ, the Laplacian operator for (M, g), which has 0 and n as the indicial roots. The argument in [10] shows that $f^1 = O(\rho^n)$. $\square$

We can apply lemma 4.19 to weaken the decay assumption in theorem 4.9.

References

[1] L. Ahlfors, *Mobiüs transformations in several dimensions,* Ordway Professorship Lectures in Mathematics, University of Minnesota, School of Mathematics.

[2] P. Aviles, H.I. Choi & M. Micallef, *The Dirichlet problem and Fatou's theorem for harmonic maps,* J. Func. Anal. **99** (1991), 293-331.

[3] J. Eells & L. Lemaire, *Selected topics in harmonic maps,* C.B.M.S. Regional Conf. Series 50, A.M.S., Providence, R.I., 1983.

[4] J. Eells & L. Lemaire: *Another report on harmonic maps,* Bull. London Math. Soc. **20** (1988), 385-524.

[5] W.D. Garber, S.N.M. Ruijsenaars, E. Seiler & D. Burns, *On finite action solutions of the nonlinear σ-model,* Ann. of Phys. **119** (1979), 305-325.

[6] R. Graham & J. Lee, *Einstein metrics with prescribed conformal infinity on the ball,* Advances in Math. **87** (1991), 186-225.

[7] Z. Jin, *Liouville theorems for harmonic maps,* Invent. Math. (1991), 1-10.

[8] H. Karcher & J. Wood, *Non-existence results and growth properties for harmonic maps and forms,* J. Reine Angew. Math. **353** (1984), 165-180. maps of

[9] R. Mazzeo, *Hodge cohomology of negatively curved manifolds,* Thesis, MIT, 1986.

[10] R. Mazzeo, *Regularity for the singular Yamabe problem,* Indiana University Mathematics Journal **40** (1991), 1277-1299.

[11] R. Mazzeo & R. Melrose, *Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature,* J. Func. Anal. **75** (1987), 260-310.

[12] P. Price, *A monotonicity formula for Yang-Mills fields,* Manuscripta Math. **43** (1983), 131-166.

[13] H. Sealey, *Some conditions ensuring the vanishing of harmonic differential forms with applications to harmonic maps and Yang-Mills theory,* Math. Proc. Camb. Phil. Soc. **91** (1982), 441-452.

[14] Y.L. Xin, *Liouville type theorems and regularity of harmonic maps,* Differential Geometry and Differential Equations, (Proceedings, Shanghai, 1985), Lecture Notes in Math. 1255, Springer-Verlag, Berlin, 1987, pp. 199-208.